Published by Institute of Physics Publishing for SISSA/ISAS
Received: *July 21, 2003*
Accepted: *October 6, 2003*

On the chiral ring of $\mathcal{N} = 1$ supersymmetric gauge theories

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Abstract: We consider the chiral ring of the pure $\mathcal{N} = 1$ supersymmetric gauge theory with $\mathrm{SU}(N)$ gauge group and show that the classical relation $S_{\rm cl}^{N^2} = 0$ is modified to the exact quantum relation $(S^N - \Lambda^{3N})^N = 0$.

Keywords: String Duality, Supersymmetric Effective Theories.

Recently, much attention has been devoted to the study of four dimensional $\mathcal{N} = 1$ supersymmetric gauge theories, due to the gauge theory/matrix model correspondence [1]. This result has been clarified from the field theoretical point of view in [2, 3] by considering the chiral ring of the gauge theory and a generalization of the Konishi anomaly [4]. Furthermore, the relation with $\mathcal{N} = 2$ supersymmetric gauge theories led to an underlying duality in the $\mathcal{N} = 1$ theory [5]. This duality is strictly related to scaling properties of the matrix model free energy. The latter has been also useful in investigating the exact structure of the free energy, leading to the appearance of new bilinear terms depending on an odd integer [6]. In this respect we note that, in particular theories, there are interesting questions concerning the contributions at order S^h, with h the dual Coxeter number [7].

In this note we will consider the chiral ring structure for pure $\mathcal{N} = 1$ SU(N) gauge theory and we will argue that the classical relation [2]

$$S_{\text{cl}}^{N^2} = 0\,, \tag{1}$$

where $S = -\frac{1}{32\pi^2}\text{Tr}W_\alpha W^\alpha$ is the glueball superfield, gets modified to the exact quantum relation

$$\left(S^N - \Lambda^{3N}\right)^N = 0\,. \tag{2}$$

Let us consider the gaugino condensate in the case of SU(N). Properties of the trace for SU(N) show that classically also the following relation [2]

$$S_{\text{cl}}^N = \{\overline{Q}_{\dot\alpha}, X^{\dot\alpha}\}\,, \tag{3}$$

holds, whose generalization to gauge groups Sp(N) and SO(N) has been derived by Witten in [3]. Subsequently, in [8] it has been verified the structure of the classical ring for the exceptional Lie group G_2 conjectured in [3]. Instantons modify this relation to the exact operator relation [2]

$$S^N = \Lambda^{3N} + \{\overline{Q}_{\dot\alpha}, X^{\dot\alpha}\}\,, \tag{4}$$

that generalizes to other groups [3]. Similarly, also $S^{N^2} = 0$ receives instanton corrections. In particular, consistency with the above finding implies that there is the exact operator relation

$$\mathcal{P}(S^N, \Lambda^{3N}) = 0\,, \tag{5}$$

where $\mathcal{P} \equiv (S^N - \Lambda^{3N})P(S^N, \Lambda^{3N})$ with P a homogeneous polynomial of degree $N-1$ with a non-zero coefficient of $(S^N)^{N-1}$, whose precise form is unknown [2].

Fermi statistics requires attention in studying the quantum properties of S and its powers as these need to be defined by point splitting. Instanton calculations indicate that one may obtain a well defined field constructed out of S. For SU(N) one obtains (see [9] for a recent discussion)

$$\langle S^N \rangle = \Lambda^{3N}\,. \tag{6}$$

This result needs to be specified. There are two ways to calculate the gluino condensate. One is based on the weak-coupling instanton (WCI) calculations, giving the above result, whereas with the strong-coupling instanton (SCI) calculations the right hand side of (6) is

JHEP10(2003)011

replaced by $2[(N-1)!(3N-1)]^{-1/N}\Lambda^{3N}$. However, it turns out that cluster decomposition does not hold in the SCI [10]. Furthermore, it has been observed that on $\mathbb{R}^3\times S^1$ the results do not depend on the radius of S^1, so that one ends up with $\mathbb{R}^4$ in the infinite radius limit, or more precisely $\mathbb{R}^3\times\hat{\mathbb{R}}$, where $\hat{\mathbb{R}}\doteq\mathbb{R}\cup\{\infty\}$ is the one point compactification of $\mathbb{R}$. It is interesting to note that, in the case of $\mathcal{N}=2$, it has been shown that the instanton moduli space admits a compactification induced by the noncommutative theory. This property is intrinsic to $\mathcal{N}=2$, and so, even if the field content in the strong coupling region has not emerged yet, it should have a counterpart in a field theoretic derivation of the expansion of the dual SW prepotential $\mathcal{F}_D$ near the critical points $u=\pm\Lambda_{SW}^2$. On the other hand, this region is the one where breaking the $\mathcal{N}=2$ theory one obtains the $\mathcal{N}=1$ result (6). This would suggest that the equivalence between the calculation of the gluino condensate in $\mathbb{R}^4$, made by the WCI calculations, and the evaluation of the gluino condensate in $\mathbb{R}^3\times S^1$, may be connected with the compactification, induced by noncommutative geometry, of the instanton moduli spaces in $\mathcal{N}=2$. In this respect, it is worth recalling that in the infrared regime low-energy dynamics of noncommutative $\mathcal{N}=2$ supersymmetric U(N) Yang-Mills theories, the U(1) decouples and the SU(N) is described by the commutative SW theory [11].

Using the recursion relations for the instanton contributions, a Deligne-Knudsen-Mumford (DKM) like compactification of instanton moduli space was derived in [12]. A remarkable property of the DKM stable compactification is a sort of regularization as punctures never collide in the degeneration limit. This is at the basis of the recursive structure. It also turns out that, in a different approach, the existence of a nilpotent fermionic symmetry implies a BRST operator that leads to localization of integrals on the instanton moduli spaces [13]. This is related to Nekrasov's project, formulated in a series of papers [14, 15]. He and collaborators considered localization onto the instanton moduli space, and introduced both the Q-operator, which makes use of spacetime rotations in addition to the global gauge transformations, as well as the concept of the noncommutative instantons. Remarkably, this culminated in [16], with the explicit evaluation, for any classical gauge group, of the instanton moduli space integrals (see also [17, 18]).

The above discussion naturally leads to consider nonperturbative configurations of $\mathcal{N}=1$ supersymmetric gauge theory on $\mathbb{R}^3\times\hat{\mathbb{R}}$ rather than $\mathbb{R}^4$. As argued by Witten [3], this essentially avoids possible infrared divergences due to the fact that the calculations are performed by choosing a perturbative vacuum which is different from the true one. This one-point compactification can be seen as a way to impose boundary conditions on nonperturbative configurations rather than a change of topology of the space where the gauge theory lives. This is of interest for the definition of the gluino condensate.

We also note that the point splitting is a quantum operation which leads to a modification of the classical operator definition. Even if this operation is sensible to UV physics, the above remarks indicate that topological properties of the space-time and the possible connection to noncommutative geometry may lead to some UV/IR mixing related to the underlying fermionic nature of the gaugino condensate. Moreover, there are some analogies between the matrix model formulation and the noncommutative theory.

JHEP10(2003)011

Let us go back to the analysis of the chiral ring. The above discussion shows that we can use instanton results in order to define

$$\mathcal{O}_\Lambda \doteq S^N - \lim_{\substack{x_i \to x_j \\ \forall ij}} \langle S(x_1) \ldots S(x_N) \rangle = S^N - \Lambda^{3N}. \tag{7}$$

Note that the $X^{\dot\alpha}$ in (3) and (4) can differ only by a chiral operator: dimensional analysis and R-symmetry forbid the existence of terms $\{\overline{Q}_{\dot\alpha}, \delta X^{\dot\alpha}\}$ that vanish as $\Lambda \to 0$. The correction from $S_{\rm cl}^N$ to S^N concerns a redefinition of the glueball superfield and not $\{\overline{Q}_{\dot\alpha}, X^{\dot\alpha}\}$, that is

$$S_{\rm cl}^N = S^N - \Lambda^{3N}. \tag{8}$$

Therefore, the basic observation is that it is the N-th power of the glueball superfield that gets quantum corrections. For these reasons we used the notation $\mathcal{O}_\Lambda$ in (7) instead of $S_{\rm cl}^N$. However, since $S_{\rm cl}^{N^2} = 0$ was derived as an identity, and since, as we said, $\{\overline{Q}_{\dot\alpha}, X^{\dot\alpha}\}$ does not receive quantum corrections, it follows by (1) and (3)

$$\{\overline{Q}_{\dot\alpha}, X^{\dot\alpha}\}^N = 0. \tag{9}$$

On the other hand, being

$$\mathcal{O}_\Lambda = \{\overline{Q}_{\dot\alpha}, X^{\dot\alpha}\}, \tag{10}$$

we have

$$\mathcal{O}_\Lambda^N = 0, \tag{11}$$

that is eq. (2), as promised.

We conclude this note by observing that the emerging structure is reminiscent of the property of forms in a $(N-1)$-dimensional space. To realize the similarity let us write

$$\omega \doteq \{\overline{Q}_{\dot\alpha}, X^{\dot\alpha}\}, \tag{12}$$

where ω is a one-form on a $(N-1)$-dimensional space. Then

$$\{\overline{Q}_{\dot\alpha}, X^{\dot\alpha}\}^N = \wedge_{k=1}^N \omega = 0, \tag{13}$$

leading to a structure which should be further investigated.

Acknowledgments

It is pleasure to thank L. Alday, M. Cirafici, M. Tsulaia and G. Travaglini for discussions. Work partially supported by the European Community's Human Potential Programme under contract HPRN-CT-2000-00131 Quantum Spacetime.

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JHEP10(2003)011

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